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Pointwise supercloseness of the displacement for tensor-product quadratic pentahedral finite elements[☆]

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ABSTRACT

In this work, we derive a weak estimate of the second type for tensor-product quadratic pentahedral finite elements over uniform partitions of the domain for the Poisson equation. Combining with an estimate for the $W^{2,1}$ -seminorm of the discrete Green's function, pointwise supercloseness of the displacement between the finite element approximation and the interpolant to the true solution is given.

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1. Introduction and preliminaries

There have been many studies concerned with superconvergence of the finite element method in 3D. Books and survey papers have been published. For the literature, we refer the reader to [1–22] and references therein. Superconvergence of prismatic elements (pentahedral elements) was studied in [5,9,12,17,21]. Recently, for pointwise superconvergence of the function value (the so-called displacement), we obtained a result for block finite elements (see [13]). The present work will focus on pointwise supercloseness of the displacement for tensor-product quadratic pentahedral elements.

In this work, we shall use the symbol C to denote a generic constant, which is independent of the discretization parameter h and which may not be the same in each occurrence, and also use the standard notation for the Sobolev spaces and their norms.

The model problem considered is

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.1)$$

Here $f \in L^2(\Omega)$, and $\Omega = \Omega_{xy} \times \Omega_z \equiv (0, 1)^2 \times (0, 1) \subset \mathcal{R}^3$ is a rectangular block with a boundary, $\partial\Omega$, consisting of faces parallel to the x -, y -, and z -axes.

To discretize the problem, one proceeds as follows. The domain Ω is firstly partitioned into subcubes of side h , and each of these is then subdivided into two pentahedra (triangular prisms). We denote by $\{\mathcal{T}^h\}$ a uniform family of pentahedral partitions as above. Thus $\bar{\Omega} = \bigcup_{e \in \mathcal{T}^h} \bar{e}$. Obviously, we may write $e = D \times L$ (see Fig. 1), where D and L are open, and stand for a triangle parallel to the xy -plane and a one-dimensional interval parallel to the z -axis, respectively.

We introduce the tensor-product quadratic polynomial space denoted by \mathcal{P} , that is,

$$q(x, y, z) = \sum_{(i,j,k) \in \mathcal{I}} a_{ijk} x^i y^j z^k, \quad a_{ijk} \in \mathcal{R}, \quad q \in \mathcal{P},$$

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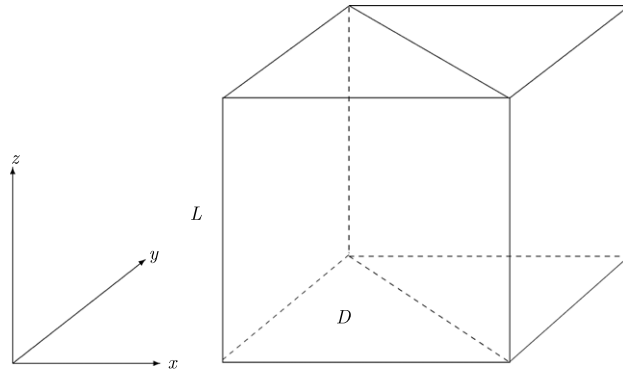


Fig. 1. Pentahedral elements.

where $\mathcal{P} = \mathcal{P}_{xy} \otimes \mathcal{P}_z$, and \mathcal{P}_{xy} and \mathcal{P}_z denote the quadratic polynomial spaces with respect to (x, y) and z , respectively. The indexing set \mathcal{I} satisfies

$$\mathcal{I} = \{(i, j, k) | i, j, k \geq 0, i + j \leq 2, k \leq 2\}.$$

Define the tensor-product quadratic interpolation operator by $\Pi^e : H_0^1(e) \rightarrow \mathcal{P}(e)$. Obviously,

$$\Pi^e = \Pi_{xy}^e \otimes \Pi_z^e = \Pi_z^e \otimes \Pi_{xy}^e,$$

where Π_{xy}^e and Π_z^e stand for the quadratic interpolation operators with respect to $(x, y) \in D$ and $z \in L$, respectively.

The weak form of problem (1.1) is

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) \equiv (\nabla u, \nabla v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx dy dz, \quad (f, v) = \int_{\Omega} f v \, dx dy dz.$$

Define the tensor-product quadratic pentahedral finite element space by

$$S_0^h(\Omega) = \{v \in H_0^1(\Omega) : v|_e \in \mathcal{P}(e) \, \forall e \in \mathcal{T}^h\}. \quad (1.2)$$

Thus, in the finite element method we find $u_h \in S_0^h(\Omega)$ such that

$$a(u_h, v) = (f, v) \quad \forall v \in S_0^h(\Omega).$$

In addition, from the definitions of Π^e and $S_0^h(\Omega)$, we can define the global interpolation operator $\Pi : H_0^1(\Omega) \rightarrow S_0^h(\Omega)$. Here $(\Pi u)|_e = \Pi^e u$.

2. A weak estimate of the second type and supercloseness

Lemma 2.1. Let $\{\mathcal{T}^h\}$ be a uniform family of pentahedral partitions of Ω , $u \in W^{5,\infty}(\Omega) \cap H_0^1(\Omega)$, and $v \in S_0^h(\Omega)$. Then, the interpolation operator Π satisfies the following weak estimate:

$$|a(u - \Pi u, v)| \leq Ch^4 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h, \quad (2.1)$$

where $|v|_{2,1,\Omega}^h = \sum_{e \in \mathcal{T}^h} |v|_{2,1,e}$.

Proof. Obviously, the interpolation remainder is

$$\begin{aligned} R &= u - \Pi u \\ &= (u - \Pi_{xy} u) + (u - \Pi_z u) + (\Pi_{xy}(u - \Pi_z u) - (u - \Pi_z u)) \\ &= R_{xy} + R_z + R^*, \end{aligned} \quad (2.2)$$

where $(\Pi_{xy} u)|_e = \Pi_{xy}^e u$, $(\Pi_z u)|_e = \Pi_z^e u$, and R^* is a high-order term. Thus, it suffices to analyze R_{xy} and R_z . We first bound the integral

$$\int_{\Omega} \nabla R_{xy} \cdot \nabla v \, dx dy dz = \int_{\Omega} (\partial_x R_{xy} \partial_x v + \partial_y R_{xy} \partial_y v) \, dx dy dz + \int_{\Omega} \partial_z R_{xy} \partial_z v \, dx dy dz = I_{xy} + I_z.$$

By the weak estimate of triangular quadratic elements [23], we have

$$|I_{xy}| \leq \int_{\Omega_z} \left| \int_{\Omega_{xy}} (\partial_x R_{xy} \partial_x v + \partial_y R_{xy} \partial_y v) dx dy \right| dz \leq Ch^4 \int_{\Omega_z} \|u\|_{4,\infty,\Omega_{xy}} |v|_{2,1,\Omega_{xy}}^h dz.$$

Thus,

$$|I_{xy}| \leq Ch^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \quad (2.3)$$

Let $S_{0,2}^h(\Omega_{xy})$ be the triangular quadratic finite element space in the domain Ω_{xy} , and $\{\psi_j\}$ the basis of this space. Obviously, the support S_j of ψ_j equals the patch of elements that share an internal edge or an internal node. Moreover, if the partition of the domain is uniform, each S_j is point-symmetric. Then for all cubic polynomials p on the support S_j of ψ_j we have

$$\int_{S_j} (p - \Pi_{xy} p) \psi_j dx dy = 0. \quad (2.4)$$

The proof of (2.4) is similar to that of Lemma 3.2 in [3].

Since $v \in S_0^h(\Omega)$, then $\partial_z v \in S_{0,2}^h(\Omega_{xy})$. Thus, $\partial_z v = \sum_j \alpha_j(z) \psi_j(x, y) \equiv \sum_j \alpha_j \psi_j$. From (2.4), we then have

$$\begin{aligned} |I_z| &= \left| \int_{\Omega} (\partial_z u - \Pi_{xy}(\partial_z u)) \partial_z v dx dy dz \right| \\ &\leq \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{\Omega_{xy}} (\partial_z u - \Pi_{xy}(\partial_z u)) \psi_j dx dy \right| dz \\ &= \int_{\Omega_z} \sum_j |\alpha_j| \left| \int_{S_j} (\partial_z u - p - \Pi_{xy}(\partial_z u - p)) \psi_j dx dy \right| dz. \end{aligned}$$

Set

$$M = \max_j \max_{z \in \Omega_z} \left| \int_{S_j} (\partial_z u - p - \Pi_{xy}(\partial_z u - p)) \psi_j dx dy \right|.$$

Taking the infimum over all $p \in \mathcal{P}^3(S_j)$, using the interpolation error estimate, we get

$$M \leq Ch^6 \|u\|_{5,\infty,\Omega}. \quad (2.5)$$

Now we define an affine transformation by $F: \hat{P} \in \hat{D} \longrightarrow P = B\hat{P} + b \in D$ such that $D = F(\hat{D})$, where $B = (b_{ij})$ is a matrix of order 2×2 . For all $w \in L^2(D)$, we write $\hat{w}(\hat{P}) = w(F\hat{P})$. The usual rules of transformation between elements and reference elements (see [3], [24, p. 40] and [23, p. 79]) tell us that there exists a constant C independent of the mesh parameters such that

$$|\hat{w}|_{0,1,\hat{D}} \leq C |\det B|^{-1} |w|_{0,1,D} \quad \text{and} \quad |\hat{w}|_{1,1,\hat{D}} \leq C \|B\| |\det B|^{-1} |w|_{1,1,D}, \quad (2.6)$$

where $\|B\| \leq Ch$ (see [3]) and $|\det B|^{-1} = \frac{\text{mes}(\hat{D})}{\text{mes}(D)} \leq Ch^{-2}$. For a fixed $z \in L = (z_{i-1}, z_i)$, we have $\partial_z v = \sum_j \alpha_j \psi_j = \sum_j \alpha_j \hat{\psi}_j = \partial_z \hat{v}$. By the equivalence of norms in the finite-dimensional space, there also exists a constant C , depending only on the reference triangle \hat{D} , such that

$$\sum_{j=1}^6 |\alpha_j| \leq C \|\hat{\partial}_z v\|_{1,1,\hat{D}} = C \left(\|\hat{\partial}_z v\|_{0,1,\hat{D}} + \|\hat{\partial}_z v\|_{1,1,\hat{D}} \right). \quad (2.7)$$

Combining (2.6) and (2.7) yields

$$\sum_{j=1}^6 |\alpha_j| \leq Ch^{-2} (|\partial_z v|_{0,1,D} + |\partial_z v|_{1,1,D}).$$

Summing over all D in the partition of the domain Ω_{xy} gives

$$\sum_j |\alpha_j| \leq Ch^{-2} (|\partial_z v|_{0,1,\Omega_{xy}} + |\partial_z v|_{1,1,\Omega_{xy}}) = Ch^{-2} \|\partial_z v\|_{1,1,\Omega_{xy}}.$$

By the Poincaré inequality, we immediately get

$$\sum_j |\alpha_j| \leq Ch^{-2} |v|_{2,1,\Omega_{xy}}^h. \quad (2.8)$$

From (2.5) and (2.8),

$$|I_z| \leq Ch^4 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h. \quad (2.9)$$

Using (2.3) and (2.9), we immediately obtain

$$\left| \int_{\Omega} \nabla R_{xy} \cdot \nabla v \, dx dy dz \right| \leq Ch^4 \|u\|_{5,\infty,\Omega} |v|_{2,1,\Omega}^h. \quad (2.10)$$

Next, we consider the integral

$$\int_{\Omega} \nabla R_z \cdot \nabla v \, dx dy dz = \int_{\Omega} (\partial_x R_z \partial_x v + \partial_y R_z \partial_y v + \partial_z R_z \partial_z v) \, dx dy dz. \quad (2.11)$$

Let $\{l_j(z)\}_{j=0}^{\infty}$ be the normalized orthogonal Legendre polynomial system from the space $\mathcal{L}^2(L)$, and $\partial_z u \in \mathcal{L}^2(L)$. For a fixed point $(x, y) \in D$, we then have the following expansion:

$$\partial_z u = \sum_{j=0}^{\infty} \mu_j(x, y) l_j(z),$$

where

$$\mu_j(x, y) = \int_L \partial_z u l_j(z) \, dz, \quad j \geq 0.$$

Set

$$\omega_0(z) = 1, \quad \omega_{j+1}(z) = \int_{z_{i-1}}^z l_j(\xi) \, d\xi, \quad j \geq 0.$$

Then we have

$$u(x, y, z) = \sum_{j=0}^{\infty} \beta_j(x, y) \omega_j(z), \quad (x, y, z) \in e = D \times L, \quad (2.12)$$

where

$$\beta_0(x, y) = u(x, y, z_{i-1}), \quad \beta_j(x, y) = \int_L \partial_z u l_{j-1}(z) \, dz = O(h^{j-\frac{1}{2}}), \quad j \geq 1. \quad (2.13)$$

Let Π_z^e be the quadratic interpolation operator of projection type with respect to z defined by

$$\Pi_z^e u = \sum_{j=0}^2 \beta_j(x, y) \omega_j(z), \quad (x, y, z) \in e = D \times L. \quad (2.14)$$

Thus, the interpolation remainder is

$$R_z = u - \Pi_z^e u = \sum_{j=3}^{\infty} \beta_j(x, y) \omega_j(z), \quad (x, y, z) \in e.$$

To bound the integral (2.11), it suffices to consider the main term $r_3 = \beta_3(x, y) \omega_3(z)$. For every $v \in S_0^h(\Omega)$,

$$\begin{aligned} \left| \int_e \partial_x r_3 \partial_x v \, dx dy dz \right| &= \left| \int_e \left(\int_L \partial_x \partial_z u(x, y, z) l_2(z) \, dz \right) \omega_3(z) \partial_x v \, dx dy dz \right| \\ &= \left| \int_e \left(\int_L \partial_x \partial_z^3 u(x, y, z) D^{-2} l_2(z) \, dz \right) D^{-1} \omega_3(z) \partial_z \partial_x v \, dx dy dz \right| \\ &\leq Ch^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,e}, \end{aligned}$$

where $\frac{d^2(D^{-2}l_2(z))}{dz^2} = l_2(z)$ and $\frac{d(D^{-1}\omega_3(z))}{dz} = \omega_3(z)$.

Thus,

$$\left| \int_e \partial_x R_z \partial_x v \, dx dy dz \right| \leq Ch^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,e}. \quad (2.15)$$

Similarly,

$$\left| \int_e \partial_y R_z \partial_y v \, dx dy dz \right| \leq Ch^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,e}. \quad (2.16)$$

In addition, by the orthogonality of the Legendre polynomial system, we have

$$\int_e \partial_z r_3 \partial_z v \, dx dy dz = \int_e \beta_3(x, y) l_2(z) \partial_z v \, dx dy dz = 0.$$

Thus,

$$\int_e \partial_z R_z \partial_z v \, dx dy dz = 0. \quad (2.17)$$

From (2.15)–(2.17), summing over all elements yields

$$\left| \int_\Omega \nabla R_z \cdot \nabla v \, dx dy dz \right| \leq Ch^4 \|u\|_{4,\infty,\Omega} |v|_{2,1,\Omega}^h. \quad (2.18)$$

The result (2.1) follows immediately from (2.2), (2.10) and (2.18). \square

To analyze pointwise supercloseness, we need to introduce the discrete Green's function defined by

$$(\nabla v, \nabla G_Z^h) = v(Z) \quad \forall v \in S_0^h(\Omega). \quad (2.19)$$

With complicated arguments, we may obtain the following lemma.

Lemma 2.2. For $G_Z^h \in S_0^h(\Omega)$ the discrete Green's function, we have the following estimate:

$$|G_Z^h|_{2,1,\Omega}^h \leq C |\ln h|^{\frac{2}{3}}. \quad (2.20)$$

The proof of Lemma 2.2 can be found in [13].

From (2.1), (2.19) and (2.20), we immediately obtain the following theorem.

Theorem 2.1 (Supercloseness). Let $\{\mathcal{T}^h\}$ be a uniform family of pentahedral partitions of Ω , and $u \in W^{5,\infty}(\Omega) \cap H_0^1(\Omega)$. For u_h and Πu , the tensor-product quadratic pentahedral finite element approximation and the interpolant corresponding to u , respectively, we have the following supercloseness estimate:

$$|u_h - \Pi u|_{0,\infty,\Omega} \leq Ch^4 |\ln h|^{\frac{2}{3}} \|u\|_{5,\infty,\Omega}.$$

Comment. Lemma 2.1 and Theorem 2.1 are proved under a very high regularity assumption such as $u \in W^{5,\infty}(\Omega)$. Therefore, the results in the work are only academic.

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